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Irreducible decompositions of physical tensors of high orders

Q.-S. ZHENG* and W.-N. ZOU
Department of Engineering Mechanics, Tsinghua University, Beijing 100084, China; e-mail: zhengqs@tsinghua.edu.cn

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Abstract. In this paper we use the term 'physical tensor' to stand for a tensor that belongs to a tensor subspace. Based on the relationship among the characters of rotation representation, some techniques are developed in order to give the numbers of independent deviatoric tensors contained in the irreducible decompositions of physical tensors, even prior to the constructions of the irreducible decompositions. A number of examples are shown and many of them are new results.

Key words: irreducible decompositions, deviatoric tensors, high-order tensors, elasticity.

1. Introduction

Denote by $\mathcal{E}$ the basic (or physical) Euclidean space. Most tensors involved in the physical and engineering sciences belong to certain linear subspaces of the full tensor spaces in $\mathcal{E}$. For instance, stress, strain and thermal expansion tensors are elements of the second-order symmetric tensor space $\mathcal{S}$. A photo-elastic tensor $\mathbf{M}$ is a linear transformation on $\mathcal{S}$, or equivalently, can be read as a second-order tensor in $\mathcal{S}$, or equivalently, an element of $\mathcal{S} \otimes \mathcal{S}$, where $\otimes$ denotes tensor product. An elastic tensor $\mathbf{E}$ and a second rank elastic tensor $\mathbf{E}_2$ are second-order symmetric and third-order symmetric tensors in $\mathcal{S}$. A piezoelectric (or piezomagnetic) tensor $\mathbf{P}$ is a linear transformation from $\mathcal{S}$ to $\mathcal{E}$, or equivalently, an element of $\mathcal{E} \otimes \mathcal{S}$. In their component forms these tensors read:

$$\left.\begin{array}{ll} P_{ijk} = P_{ikj}, & E_{ijkl} = E_{jikl} = E_{klij}, \\ M_{ijkl} = M_{jikl} = M_{ijlk}, & E^2_{ijklmn} = E^2_{jiklmn} = E^2_{klijmn} = E^2_{mnklij} \end{array}\right\}.$$

Modelling a quadratic, cubic, or quartic yielding function in the theory of plasticity requires a second-, third-, or fourth-order symmetric tensor in $\mathcal{S}$, rather than a generic fourth-, sixth-, or eighth-order tensor in $\mathcal{E}$.

Therefore, we use the term *physical tensors* to represent tensors that belong to linear subspaces of the full tensor spaces in $\mathcal{E}$. The aim is to develop some specific techniques for physical tensors in analyzing or constructing their irreducible decompositions, that may not be available for generic tensors.

It is well known from the theory of group representations that, in principle, a tensor of any finite order can be decomposed into a sum of irreducible tensors. Irreducible decompositions of tensors allow to separate the tensors in consideration into their isotropic and various anisotropic parts, and distinguish the rotationally invariant quantity from every anisotropic

* To whom correspondence should be addressed.

part. Thus, this enables easier and more intuitive physical understandings and characteristics of the tensors. Proper examples of such advantages can be seen in many papers (*e.g.* [1, 2]). Irreducible decompositions may even become a necessity in formulating a physical or constitutive equation which involves high-order tensors.

Some operable methods on irreducible decompositions of tensors were proposed by Spencer [3], Hannabuss [4], Rychlewski [5] and Zou *et al.* [6] among few others. All these works deal with generic tensors, rather than physical tensors. Although the irreducible decompositions of physical tensors can be reduced from those of generic ones, such reductions need to be guided by prior knowledge on the numbers of independent deviatoric tensors that can be contained in the irreducible decompositions of the considered physical tensors. To know the number of independent deviatoric tensors in the irreducible decomposition of a concerned physical tensor prior to the detailed construction of the irreducible decomposition is itself of importance, which to a certain degree reveals the structure of this physical tensor.

To find such numbers for physical tensors of various types, prior to the construction of the irreducible decompositions, is one of the main aims of the present paper. This is done at first by linking such numbers with the characters of rotation representations as restrictions on the physical tensors and on deviatoric tensors, and then some techniques are developed for the evaluation of the characters of rotation representations for physical tensors of various types. We also develop some methods to derive the irreducible decompositions of physical tensors directly. A number of examples, many of which are new results, are given.

In the group representation theory it known that deviatoric tensors can be one-to-one connected with spherical harmonics. To make this paper self-contained, we give the relationships among deviatoric tensors, spherical harmonics and Fourier expansions in Appendix A.

Finally, we emphasize that the suggested method and obtained results in this paper are useful not only in the context of elasticity theory (*cf.* Backus [7], Cowin [8]), damage mechanics (*cf.* Onat [9], Krajcinovic and Mastilovic [10], He and Curnier [1], Zheng and Hwang [2]), plasticity and formulation of higher-order interactions among multiply physical fields (*cf.* Juresckhe [11]), but to any fields where tensors of high orders are involved.

2. Preliminaries

This paper is written in full consistence with the paper of Zou *et al.* [6], which deals with the irreducible decomposition of generic tensors. The generic nth order tensor and the generic nth order deviatoric tensor in the basic Euclidean space $\mathcal{E}$ are denoted by $\mathbf{T}^{(n)}$ or $T_{i_1\ldots i_n}$ and $\mathbf{D}^{(n)}$ or $D_{i_1\ldots i_n}$, respectively. The corresponding linear spaces are signified as $\mathcal{E}^{\otimes n}$ and $\mathcal{D}^{(n)}$, where '$\otimes$' signifies tensor product. The term *deviatoric tensor* is synonymous with *completely symmetric and traceless tensor*, *e.g.*

$$D_{i_1 i_2 i_3\ldots i_n} = D_{i_2 i_1 i_3\ldots i_n} = D_{i_3 i_2 i_1\ldots i_n} = \cdots = D_{i_n i_2 i_3\ldots i_1}, \qquad D_{ssi_3\ldots i_n} = 0.$$

By definition, scalars and vectors are zeroth- and first-order deviatoric tensors, respectively, and the generic scalar is denoted by either α or $\mathbf{D}^{(0)}$ and the generic vector, by either $\mathbf{v}$ or $\mathbf{D}^{(1)}$. It is well known that dim $\mathcal{D}^{(n)}$ is always equal to 2 for any positive integer n when dim $\mathcal{E} = 2$, and dim $\mathcal{D}^{(n)}$ equals $2n+1$ for any nonnegative integer n when dim $\mathcal{E} = 3$. Tensor components are referred to an arbitrarily given orthonormal frame $\{\mathbf{e}_i\}$ with minuscule Latin indices. The summation convention applies to repeated minuscule Latin indices only, not to majuscule ones which are also used in this paper in several cases. The inner product in $\mathcal{E}$ is denoted by a dot

'$\cdot$'. To $\mathscr{E}^{\otimes n}$ and all its subspaces (including $\mathscr{D}^{(n)}$), we appoint $T_{i_1\ldots i_n}S_{i_1\ldots i_n}$ as the inner product of any $\mathbf{T}, \mathbf{S} \in \mathscr{E}^{\otimes n}$ and use the abstract notation $\mathbf{T} \circ \mathbf{S}$ for $T_{i_1\ldots i_n}S_{i_1\ldots i_n}$.

A second-order tensor $\mathbf{R}$ is said to be *orthogonal*, if $R_{ik}R_{jk} = R_{ki}R_{kj} = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Further, if the determinant of the $\mathbf{R}$ is equal to 1, it is said to be a *rotation*. An nth-order tensor $\mathbf{I}^{(n)}$ is said to be *hemitropic* or *isotropic*, if the following invariance

$$R_{i_1j_1}\ldots R_{i_nj_n}I_{j_1\ldots j_n} = I_{i_1\ldots i_n}$$

holds for any rotation or any orthogonal tensor $\mathbf{R}$, respectively. The tensor $R_{i_1j_1}\ldots R_{i_nj_n}\,T_{j_1\ldots j_n}$ is abstractly denoted by $\mathbf{R}^{\times n}[\mathbf{T}^{(n)}]$, which induces the linear transformation $\mathbf{R}^{\times n}$ on $\mathscr{E}^{\otimes n}$, called the nth *Kronecker power* of $\mathbf{R}$ (*cf.* [12, 13]).

It is well known that the second-order identity tensor $\mathbf{1}$ is the basic isotropic tensor, and $\mathbf{1}$ and the permutation tensor $\boldsymbol{\varepsilon}$, are the basic hemitropic tensors. Because of the identities

$$\begin{aligned}
\varepsilon_{ij}\varepsilon_{rs} &= \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}, \quad \text{or}\\
\varepsilon_{ijk}\varepsilon_{rst} &= \delta_{ir}\delta_{js}\delta_{kt} + \delta_{is}\delta_{jt}\delta_{kr} + \delta_{it}\delta_{jr}\delta_{ks} - \delta_{is}\delta_{jr}\delta_{kt} - \delta_{it}\delta_{js}\delta_{kr} - \delta_{ir}\delta_{jt}\delta_{ks},
\end{aligned}$$

in two or three-dimensional space $\mathscr{E}$, respectively, any isotropic tensor is expressible as a linear combination of tensors in the form $\mathrm{per}(\mathbf{1}\otimes\cdots\otimes\mathbf{1})$, and any hemitropic tensor is expressible as a linear combination of tensors in the forms

$$\mathrm{per}(\mathbf{1}\otimes\cdots\otimes\mathbf{1}), \qquad \mathrm{per}(\boldsymbol{\varepsilon}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}), \tag{1}$$

where 'per' indicates a permutation operation. For instance, $\delta_{ir}\delta_{js}$ and $\delta_{is}\ \delta_{jr}$ are two permutations of $\delta_{ij}\delta_{rs}$.

A $\mathbf{D}^{(s)}$-*induced n*th-order *irreducible tensor* can be defined as a linear hemitropic function of $\mathbf{D}^{(s)}$, and is denoted by $\mathbf{I}^{(n,s)}\ [\mathbf{D}^{(s)}]$ or $I_{i_1\ldots i_nj_1\ldots j_s}D_{j_1\ldots j_s}$. By definition, the relation

$$\left.\begin{aligned}
&\mathbf{I}^{(n,s)}[\mathbf{R}^{\times s}[\mathbf{D}^{(s)}]] = \mathbf{R}^{\times n}[\mathbf{I}^{(n,s)}[\mathbf{D}^{(s)}]], \quad \text{or}\\
&I_{i_1\ldots i_nj_1\ldots j_s}R_{j_1k_1}\ldots R_{j_sk_s}D_{k_1\ldots k_s} = R_{i_1l_1}\ldots R_{i_nl_n}I_{l_1\ldots l_nj_1\ldots j_s}D_{j_1\ldots j_s},
\end{aligned}\right\} \tag{2}$$

holds for any rotation $\mathbf{R}$ and any sth-order deviatoric tensor $\mathbf{D}^{(s)}$. This requires that $I_{i_1\ldots i_nj_1\ldots j_s}$ or, in brief, $\mathbf{I}^{(n,s)}$ must be $(n+s)$th-order hemitropic tensor.

In the work of Zou *et al.*, the recursive formulae for constructing the orthogonal irreducible decompositions of the generic nth order tensor $\mathbf{T}^{(n)}$ for any finite positive integer n in both two and three dimensions are established. The decompositions take the form

$$\mathbf{T}^{(n)} = \sum_{J=1}^{J_0^n}\alpha_J\mathbf{I}_J^{(n,0)} + \sum_{J=1}^{J_1^n}\mathbf{I}_J^{(n,1)}[\mathbf{v}_J] + \sum_{J=1}^{J_2^n}\mathbf{I}_J^{(n,2)}[\mathbf{D}_J^{(2)}] + \cdots + \sum_{J=1}^{J_n^n}\mathbf{I}_J^{(n,n)}[\mathbf{D}_J^{(n)}], \tag{3}$$

or in brief,

$$\mathbf{T}^{(n)} = J_0^n\alpha \dotplus J_1^n\mathbf{v} \dotplus J_2^n\mathbf{D}^{(2)} \dotplus \cdots \dotplus J_n^n\mathbf{D}^{(n)}, \tag{4}$$

with

$$J_s^n = \begin{cases} 0, & \text{if } n+s = odd, \\ \binom{n}{\frac{n+s}{2}}, & \text{if } n+s = even, \end{cases} \tag{5}$$

$$J_s^n = \begin{bmatrix} n \\ s \end{bmatrix} - \begin{bmatrix} n \\ s+1, \end{bmatrix}, \tag{6}$$

in two or three dimensions, respectively, where $\binom{n}{s}$, $[{n \atop s}]$ and J_s^n in two dimensions are the coefficients of x^s in the expansions of $(1+x)^n$, $(1+x+1/x)^n$ and $(x+1/x)^n$, respectively. The majuscule Latin index J $(J = 1, \ldots, J_s^n)$ signifies that the decomposition (3) involves J_s^n *independent* sth-order deviatoric tensors. In other words, J_s^n calculated from (5) or (6) denotes the number of independent sth-order deviatoric tensors involved in the decompositions (3) or (4).

A given irreducible decomposition (3) means that the set of hemitropic tensors $\mathbf{I}_J^{(n,s)}$ is given or independent of $\mathbf{T}^{(n)}$. One can easily verify that

$$1 + 2\sum_{s=1}^{n} J_s^n = 2^n \quad \text{or} \quad \sum_{s=0}^{n} (1+2s) J_s^n = 3^n \tag{7}$$

in two or three dimensions, respectively. This implies the following basic proposition.

PROPOSITION 1. *All the linear transformations* $\mathbf{I}_J^{(n,s)}\colon \mathcal{D}^{(s)} \to \mathcal{E}^{\otimes n}$ *in the irreducible decomposition* (3) *must be isomorphic, that is,* $\dim \mathbf{I}_J^{(n,s)}[\mathcal{D}^{(s)}] = \dim \mathcal{D}^{(s)}$.

3. Numbers of deviatoric tensors and characters of rotation representations

3.1. Kronecker products and characters of linear transformations

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two Euclidean spaces and $\mathbf{L}_1$ and $\mathbf{L}_2$ be any two linear transformations on $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. We always assemble the inner product

$$(\mathbf{v}_1 \otimes \mathbf{v}_2) \circ (\mathbf{u}_1 \otimes \mathbf{u}_2) = (\mathbf{v}_1 \circ \mathbf{u}_1)(\mathbf{v}_2 \circ \mathbf{u}_2), \quad (\text{for any } \mathbf{v}_k, \mathbf{u}_k \in \mathcal{V}_k)$$

to the tensor product space $\mathcal{V}_1 \otimes \mathcal{V}_2$, so that $\mathcal{V}_1 \otimes \mathcal{V}_2$ becomes a new Euclidean space of the dimensions $\dim \mathcal{V}_1 \dim \mathcal{V}_2$. The Kronecker product (*cf.* [13, pp. 68–70]), denoted by $\mathbf{L}_1 \times \mathbf{L}_2$, of $\mathbf{L}_1$ and $\mathbf{L}_2$, is defined as a linear transformation on $\mathcal{V}_1 \otimes \mathcal{V}_2$ such that

$$\mathbf{L}_1 \times \mathbf{L}_2[\mathbf{v}_1 \otimes \mathbf{v}_2] = \mathbf{L}_1[\mathbf{v}_1] \otimes \mathbf{L}_2[\mathbf{v}_2], \quad (\text{for any } \mathbf{v}_k \in \mathcal{V}_k). \tag{8}$$

Thus, if σ_k for $k = 1, 2$ are (real or complex) eigenvalues of $\mathbf{L}_k$ with respect to $\mathbf{v}_k$ the associated eigentensors, *i.e.* $\mathbf{L}_1[\mathbf{v}_1] = \sigma_1 \mathbf{v}_1$ and $\mathbf{L}_2[\mathbf{v}_2] = \sigma_2\, \mathbf{v}_2$, then $\sigma_1\sigma_2$ is an eigenvalue of $\mathbf{L}_1 \times \mathbf{L}_2$ with $\mathbf{v}_1 \otimes \mathbf{v}_2$ as an associated eigentensor, and the set of eigenvalues of $\mathbf{L}_1 \times \mathbf{L}_2$ is

$$\{\sigma_1\sigma_2\colon \text{for all eigenvalues } \sigma_1 \text{ of } \mathbf{L}_1 \text{ and } \sigma_2 \text{ of } \mathbf{L}_2\}. \tag{9}$$

The apparent associativity of the Kronecker product allows us to define Kronecker powers recursively, such as the nth Kronecker power $\mathbf{R}^{\times n} = \mathbf{R} \times \mathbf{R}^{\times(n-1)}$ of the rotation $\mathbf{R}$ which was described previously.

In the theory of group representations, the concept of *character* $\chi(\mathbf{L})$ of a linear transformation $\mathbf{L}$ plays a key role. $\chi(\mathbf{L})$ is defined as the sum of all eigenvalues of $\mathbf{L}$. From (9) it is seen that

$$\chi(\mathbf{L}_1 \times \mathbf{L}_2) = \chi(\mathbf{L}_1)\chi(\mathbf{L}_2). \tag{10}$$

Particularly, for any rotation $\mathbf{R}$ and any positive integer n, $\chi(\mathbf{R}^{\times n}) = \chi(\mathbf{R})^n$.

It is well known that in two dimensions, the eigenvalues of $\mathbf{R}$ are two unit conjugated complex numbers $\mathrm{e}^{\pm\,\mathrm{i}\theta}$, say, so that we have

$$\chi(\mathbf{R}) = \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} = 2\cos\theta, \qquad \chi(\mathbf{R}^{\times n}) = (2\cos\theta)^n; \tag{11}$$

and in three dimensions, the set of eigenvalues of $\mathbf{R}$ can be expressed as $\{1, \mathrm{e}^{\pm\,\mathrm{i}\theta}\}$, so that we have

$$\chi(\mathbf{R}) = 1 + \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} = 1 + 2\cos\theta, \qquad \chi(\mathbf{R}^{\times n}) = (1 + 2\cos\theta)^n, \tag{12}$$

where $\mathrm{i} = \sqrt{-1}$ is the unit imaginary number. Geometrically, θ is the *rotation angle* of $\mathbf{R}$.

3.2. Numbers of independent deviatoric tensors in linking characters of rotation representations

It is easy to check that for any rotation $\mathbf{R}$ and any positive integer n, $\mathbf{R}^{\times n}[\mathbf{D}^{(n)}]$ is still deviatoric. In other words, the linear space $\mathcal{D}^{(n)}$ is an invariant linear subspace of $\mathbf{R}^{\times n}$, *i.e.* $\mathbf{R}^{\times n}[\mathcal{D}^{(n)}] \subseteq \mathcal{D}^{(n)}$. Therefore, we can define the restriction, denoted by $\mathbf{R}^{\times n}|_{\mathcal{D}^{(n)}}$ or the simpler $\mathbf{R}^{\times n}|_{\mathcal{D}}$, of $\mathbf{R}^{\times n}$ on $\mathcal{D}^{(n)}$ through the relation $\mathbf{R}^{\times n}[\mathbf{D}^{(n)}] = \mathbf{R}^{\times n}|_{\mathcal{D}}[\mathbf{D}^{(n)}]$. Noting the identity (2) for any $\mathbf{D}^{(s)}$-induced nth-order irreducible tensor, we see that the $\mathcal{D}^{(s)}$-induced linear space $\mathbf{I}^{(n,s)}[\mathcal{D}^{(s)}]$ is also an invariant linear subspace of $\mathbf{R}^{\times n}$. Furthermore, from (2) one can see that, if σ is an eigenvalue of $\mathbf{R}^{\times s}|_{\mathcal{D}}$, then it is also an eigenvalue of the restriction of $\mathbf{R}^{\times n}$ on $\mathbf{I}^{(n,s)}[\mathcal{D}^{(s)}]$.

Now, we are ready to establish the important relationship between the characters of rotation representations and the numbers of independent deviatoric tensors in the irreducible decompositions of physical tensors. Consider the linear space $\mathcal{M}^{(n)}$ of all nth-order tensors in a certain physical tensor type (*e.g.* the elastic tensor type, piezoelectric tensor type). A general element in $\mathcal{M}^{(n)}$ will be denoted by $\mathbf{M}^{(n)}$. Restricting the irreducible decomposition (3) on $\mathcal{M}^{(n)}$, we are led to the irreducible decomposition of $\mathbf{M}^{(n)}$, which is expressed coincidently in the normal and brief forms:

$$\begin{aligned}\mathbf{M}^{(n)} &= \sum_{J=1}^{j_0^n} \alpha_J \mathbf{I}_J^{(n,0)} + \sum_{J=1}^{j_1^n} \mathbf{I}_J^{(n,1)}[\mathbf{v}_J] + \sum_{J=1}^{j_2^n} \mathbf{I}_J^{(n,2)}[\mathbf{D}_J^{(2)}] + \cdots + \sum_{J=1}^{j_n^n} \mathbf{I}_J^{(n,n)}[\mathbf{D}_J^{(n)}] \\ &= j_0^n \alpha \dotplus j_1^n \mathbf{v} \dotplus j_2^n \mathbf{D}^{(2)} \dotplus \cdots \dotplus j_n^n \mathbf{D}^{(n)}.\end{aligned} \tag{13}$$

Recalling Proposition 1, we observe that all the spaces $\mathbf{I}_J^{(n,s)}[\mathcal{D}_J^{(s)}]$ for $J = 1, \ldots, j_s^n$ are of the same dimensions as $\dim \mathcal{D}^{(s)}$. Consequently, the restriction of $\mathbf{R}^{\times n}$ on $\mathbf{I}_J^{(n,s)}[\mathcal{D}^{(s)}]$ is equivalent to $\mathbf{R}^{\times s}|_{\mathscr{D}}$. These analyses thus yield that the eigenvalue set and the character of $\mathbf{R}^{\times n}$ as restriction on $\mathbf{I}_J^{(n,s)}[\mathscr{D}^{(s)}]$ are completely the same as those of $\mathbf{R}^{\times s}|_{\mathscr{D}}$. We obtain the following basic counting number property.

PROPOSITION 2. *The numbers* $j_0^n, j_1^n, \ldots, j_n^n$ *of independent deviatoric tensors in the orthogonal irreducible decomposition* (13) *and the characters are related in the form*

$$\chi(\mathbf{R}^{\times n}|_{\mathscr{M}^{(n)}}) = j_0^n + j_1^n \chi(\mathbf{R}^{\times 1}|_{\mathscr{D}}) + \cdots + j_n^n \chi(\mathbf{R}^{\times n}|_{\mathscr{D}}). \tag{14}$$

It is well known in the theory of group representations, or can be seen in the next subsection for the sake of making this paper self-contained, that the set of eigenvalues and character of $\mathbf{R}^{\times s}|_{\mathscr{D}}$ are respectively

$$\{e^{\pm is\theta}\}, \quad \chi(\mathbf{R}^{\times s}|_{\mathscr{D}}) = 2\cos s\theta \tag{15}$$

in two dimensions, and

$$\{1, e^{\pm i\theta}, e^{\pm i2\theta}, \ldots, e^{\pm is\theta}\}, \quad \chi(\mathbf{R}^{\times s}|_{\mathscr{D}}) = 1 + 2(\cos\theta + \cdots + \cos s\theta) \tag{16}$$

in three dimensions. We are ready to prove the following *counting number theorem*.

THEOREM 3. *When the character* $\chi(\mathbf{R}^{\times n}|_{\mathscr{M}^{(n)}})$ *is written in the form*

$$\begin{aligned} \chi(\mathbf{R}^{\times n}|_{\mathscr{M}^{(n)}}) &= \chi_0^n + 2(\chi_1^n \cos\theta + \chi_2^n \cos 2\theta + \cdots + \chi_n^n \cos n\theta) \\ &= \chi_0^n + \sum_{s=1}^{n} \chi_s^n (e^{is\theta} + e^{-is\theta}), \end{aligned} \tag{17}$$

the numbers j_s^n, $s = 0, 1, \ldots, n$, *of independent deviatoric tensors in the irreducible decomposition* (13) *of the generic physical tensor* $\mathbf{M}^{(n)} \in \mathscr{M}^{(n)}$ *can be calculated as follows*

$$j_s^n = \begin{cases} \chi_s^n, & \text{in } 2D \text{ space}, \\ \chi_s^n - \chi_{s+1}^n, & \text{in } 3D \text{ space, with } \chi_{n+1}^n = 0. \end{cases} \tag{18}$$

Proof. In two dimensions, substituting $(15)_2$ in (14) and comparing it with (17), we find that $(18)_1$ follows directly. In three dimensions, substitution of $(16)_2$ results in

$$\begin{aligned} \chi(\mathbf{R}^{\times n}|_{\mathscr{M}^{(n)}}) &= (j_0^n + \cdots + j_n^n) + 2(j_1^n + \cdots + j_n^n)\cos\theta \\ &\quad + \cdots + 2(j_{n-1}^n + j_n^n)\cos(n-1)\theta + 2j_n^n \cos n\theta. \end{aligned}$$

Comparing the above coefficients with those in $(17)_2$ we simply arrive at the relation $(18)_2$. □

Thus, it is possible to obtain the numbers j_s^n prior to the constructions of irreducible decompositions, and the key to obtain j_s^n is to evaluate the characters $\chi(\mathbf{R}^{\times n}|_{\mathscr{M}^{(n)}})$ for physical tensors. Now, one can skip over the next subsection, which is written mostly for the purpose of making this paper self-contained, and go to the next section which deals with some interesting types of physical tensors.

3.3. CHARACTERS OF $\mathbf{R}^{\times n}|_{\mathcal{D}}$: A SUPPLEMENT

In two dimensions, the definition of a rotation $\mathbf{R}$ can be given geometrically by

$$\mathbf{R}\cdot\mathbf{e}_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \qquad \mathbf{R}\cdot\mathbf{e}_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2,$$

or in their brief form: $\mathbf{R}\cdot\boldsymbol{\omega} = \mathrm{e}^{-\mathrm{i}\theta}\boldsymbol{\omega}$ with $\boldsymbol{\omega} = \mathbf{e}_1 + \mathrm{i}\,\mathbf{e}_2$, where $\{\mathbf{e}_i\}$ is a given orthonormal basis of $\mathcal{E}$. Consequently, for any positive integer n, we have $\mathbf{R}^{\times n}\,[\boldsymbol{\omega}^{\otimes n}] = \mathrm{e}^{-\mathrm{i}n\theta}\boldsymbol{\omega}^{\otimes n}$, or

$$\mathbf{R}^{\times n}[\mathbf{P}^{(n)}] = \cos n\theta\mathbf{P}^{(n)} + \sin n\theta\mathbf{Q}^{(n)}, \mathbf{R}^{\times n}[\mathbf{Q}^{(n)}] = -\sin n\theta\mathbf{P}^{(n)} + \cos n\theta\mathbf{Q}^{(n)}, \tag{19}$$

where $\boldsymbol{\omega}^{\otimes n}$ ($\boldsymbol{\omega}^{\otimes n} = \boldsymbol{\omega}\otimes\,\boldsymbol{\omega}^{\otimes(n-1)}$) is the nth tensor power of $\boldsymbol{\omega}$, and $\mathbf{P}^{(n)}$ and $\mathbf{Q}^{(n)}$ as the real and imaginary parts of $\boldsymbol{\omega}^{\otimes n}$ constitute an orthogonal basis of $\mathcal{D}^{(n)}$. The formulae (19) are very useful for characterizing various anisotropies ([14, 15]). A component form of $\mathbf{D}^{(n)}$ can be written as follows

$$\mathbf{D}^{(n)} = \mathfrak{Re}\{c^{(n)}\bar{\boldsymbol{\omega}}^{\otimes n}\}, \tag{20}$$

where $c^{(n)}$ is a complex number, and $\mathfrak{Re}$ indicates the real part. From (19) and (20) it is seen that the eigenvalue set of the restriction $\mathbf{R}^{\times n}|_{\mathcal{D}}$ of $\mathbf{R}^{\times n}$ on the $2D$ subspace $\mathcal{D}^n$ for each positive integer n consists of the pair $\mathrm{e}^{\pm\mathrm{i}n\theta}$ and the character is $\chi(\mathbf{R}^{\times n}|_{\mathcal{D}}) = 2\cos n\theta$, as shown in (15).

In three dimensions, for any given rotation $\mathbf{R}$, we assign an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathcal{E}$ such that $\mathbf{e}_3$ is the unit axial vector of $\mathbf{R}$, *i.e.* $\mathbf{R}\cdot\mathbf{e}_3 = \mathbf{e}_3$. Thus, a geometrical definition of $\mathbf{R}$ can be given as

$$\mathbf{R}\cdot\mathbf{e}_3 = \mathbf{e}_3, \qquad \mathbf{R}\cdot\mathbf{e}_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \qquad \mathbf{R}\cdot\mathbf{e}_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2,$$

or equivalently, $\mathbf{R}\cdot\mathbf{e}_3 = \mathbf{e}_3$ and $\mathbf{R}\cdot\boldsymbol{\omega} = \mathrm{e}^{-\mathrm{i}\theta}\boldsymbol{\omega}$. Further, introducing the complex vector

$$\boldsymbol{\omega}_\phi = 2\mathbf{e}_3 + \mathrm{e}^{-\mathrm{i}\phi}\boldsymbol{\omega} - \mathrm{e}^{\mathrm{i}\phi}\bar{\boldsymbol{\omega}} \tag{21}$$

and noting $\boldsymbol{\omega}_\phi\cdot\boldsymbol{\omega}_\phi = 0$, we see that the nth tensor power $\boldsymbol{\omega}_\phi^{\otimes n}$ is deviatoric. Consequently, the coefficient tensors $\mathbf{P}_0^n$ and $\mathbf{W}_s^{(n)}$ in the expansion of $\boldsymbol{\omega}_\phi^{\otimes n}$

$$\boldsymbol{\omega}_\phi^{\otimes n} = \mathbf{P}_0^{(n)} + \sum_{s=1}^{n}\{\mathrm{e}^{-\mathrm{i}s\phi}\mathbf{W}_s^{(n)} + (-1)^s\,\mathrm{e}^{\mathrm{i}s\phi}\bar{\mathbf{W}}_s^{(n)}\}$$

are all deviatoric. It is evident that $\mathbf{P}_0^{(n)}$ is real while $\mathbf{W}_s^{(n)}$ for $1 \leqslant s \leqslant n$ are complex. The relations

$$\mathbf{R}^{\times n}[\boldsymbol{\omega}_\phi^{\otimes n}] = \boldsymbol{\omega}_{\theta+\phi}^{\otimes n} = \mathbf{P}_0^{(n)} + \sum_{s=1}^{n}\{\mathrm{e}^{-\mathrm{i}s(\theta+\phi)}\mathbf{W}_s^{(n)} + (-1)^s\,\mathrm{e}^{\mathrm{i}s(\theta+\phi)}\bar{\mathbf{W}}_s^{(n)}\}$$

further yields

$$\mathbf{R}^{\times n}[\mathbf{P}_0^{(n)}] = \mathbf{P}_0^{(n)}, \qquad \mathbf{R}^{\times n}[\mathbf{W}_s^{(n)}] = \mathrm{e}^{-\mathrm{i}s\theta}\mathbf{W}_s^{(n)}, \quad 1 \leqslant s \leqslant n. \tag{22}$$

Therefore, the subspace $\mathcal{D}_0^n$ spanned by $\mathbf{P}_0^{(n)}$ and the subspace $\mathcal{D}_s^n$ spanned by the real and imaginary parts $\mathbf{P}_s^{(n)}$ and $\mathbf{Q}_s^{(n)}$, of $\mathbf{W}_s^{(n)}$ are all the invariant subspaces of $\mathbf{R}^{\times n}$ with one-fold the eigenvalue 1 and one-fold the pair of eigenvalues $\mathrm{e}^{\pm\mathrm{i}s\theta}$, respectively. Further, by observing

$$\mathbf{W}_s^{(n)} \circ \mathbf{R}^{\times n}[\mathbf{W}_t^{(n)}] = \mathrm{e}^{-\mathrm{i}t\theta}\mathbf{W}_s^{(n)} \circ \mathbf{W}_t^{(n)} = (\mathbf{R}^T)^{\times n}[\mathbf{W}_s^{(n)}] \circ \mathbf{W}_t^{(n)} = \mathrm{e}^{-\mathrm{i}s\theta}\mathbf{W}_s^{(n)} \circ \mathbf{W}_t^{(n)},$$

$$\mathbf{W}_s^{(n)} \circ \mathbf{R}^{\times n}[\bar{\mathbf{W}}_t^{(n)}] = \mathrm{e}^{\mathrm{i}t\theta}\mathbf{W}_s^{(n)} \circ \bar{\mathbf{W}}_t^{(n)} = (\mathbf{R}^T)^{\times n}[\mathbf{W}_s^{(n)}] \circ \bar{\mathbf{W}}_t^{(n)} = \mathrm{e}^{-\mathrm{i}s\theta}\mathbf{W}_s^{(n)} \circ \bar{\mathbf{W}}_t^{(n)},$$

where $\mathbf{R}^T$ denotes the transpose of $\mathbf{R}$, we find that the $(2n+1)$ deviatoric tensors $\mathbf{P}_0^{(n)}$, $\mathbf{P}_1^{(n)}$, $\mathbf{Q}_1^{(n)}, \ldots, \mathbf{P}_n^{(n)}$, $\mathbf{Q}_n^{(n)}$ are mutually orthogonal. Consequently, due to $\dim \mathcal{D}^n = 2n+1$ in three dimensions, these $(2n+1)$ tensors constitute an orthogonal basis of $\mathcal{D}^n$. In other words, any $\mathbf{D}^{(n)} \in \mathcal{D}^n$ can be expressed in the following component form

$$\mathbf{D}^{(n)} = \alpha^{(0)}\mathbf{P}_0^{(n)} + \mathfrak{Re}\sum_{s=1}^{n}\{c_s^{(n)}\bar{\mathbf{W}}_s^{(n)}\}, \tag{23}$$

where $\alpha^{(n)}$ is real and $c_s^{(n)}$ are complex numbers. Furthermore, from (22) and (20) it is seen that the set of eigenvalues and the character of the restriction $\mathbf{R}^{\times n}|_{\mathcal{D}}$ are just those were given in (16).

4. Irreducible decompositions of physical tensors

4.1. Physical tensors in types of $\mathcal{M}_1^{(n_1)} \otimes \mathcal{M}_2^{(n_2)}$ and $\mathcal{S}^{\otimes n}$

The following proposition and its corollary are quite practical.

PROPOSITION 4. *Let n_1 and n_2 be any two positive integers, $n = n_1 + n_2$, and $\mathcal{M}_1^{(n_1)}$ and $\mathcal{M}_2^{(n_2)}$ be two physical tensor spaces of orders n_1 and n_2, respectively. For the physical tensor space $\mathcal{M}^{(n)} = \mathcal{M}_1^{(n_1)} \otimes \mathcal{M}_2^{(n_2)}$, then*

$$\chi(\mathbf{R}^{\times n}|_{\mathcal{M}^{(n)}}) = \chi(\mathbf{R}^{\times n_1}|_{\mathcal{M}_1^{(n_1)}})\chi(\mathbf{R}^{\times n_2}|_{\mathcal{M}_2^{(n_2)}}). \tag{24}$$

COROLLARY 5. *Let $\mathcal{S}$ be the second-order symmetric tensor space in the normal Euclidean space $\mathcal{E}$, n be any positive integer, and $\mathcal{S}^{\otimes n}$ be the nth order tensor space in $\mathcal{S}$. Then,*

$$\chi(\mathbf{R}^{\times(2n)}|_{\mathcal{S}^{\otimes n}}) = \chi(\mathbf{R}^{\times 2}|_{\mathcal{S}})^n. \tag{25}$$

Proof. According to the definition (8), we have $\mathbf{R}^{\times n}|_{\mathcal{M}^{(n)}} = \mathbf{R}^{\times n_1}|_{\mathcal{M}^{(n_1)}} \times \mathbf{R}^{\times n_2}|_{\mathcal{M}^{(n_2)}}$. Applying (10) yields (24). The property (25) is a simple consequence of (24) because of the commutativity of Kronecker product. □

As an application of (24), we have

$$\chi(\mathbf{R}^{\times(n+1)}|_{\mathcal{E}\otimes\mathcal{D}^{(n)}}) = (2\cos\theta)(2\cos n\theta) = 2[\cos(n-1)\theta + \cos(n+1)\theta]$$

in two dimensions, and

$$\begin{aligned}\chi(\mathbf{R}^{\times(n+1)}|_{\mathcal{E}\otimes\mathcal{D}^{(n)}}) &= (1 + 2\cos\theta)(1 + 2\cos\theta + \cdots + 2\cos n\theta)\\ &= 3 + 6[\cos\theta + \cdots + \cos(n-1)\theta] + 4\cos n\theta + 2\cos(n+1)\theta\end{aligned}$$

in three dimensions. Recalling Theorem 3, we obtain the following result.

PROPOSOTION 6. *For any integer $n > 1$, the irreducible decompositions of the generic element $\mathbf{G}^{(n+1)}$ of $\mathcal{E} \otimes \mathcal{D}^{(n)}$ take the following forms*

$$\mathbf{G}^{(n+1)} = \begin{cases} \mathbf{D}^{(n-1)} \dot{+} \mathbf{D}^{(n+1)} & \textit{in two dimensions}, \\ \mathbf{D}^{(n-1)} \dot{+} \mathbf{D}^{(n)} \dot{+} \mathbf{D}^{(n+1)} & \textit{in three dimensions}. \end{cases} \tag{26}$$

The result $(26)_2$ was used in the work of Zou *et al.* [6] in constructing their recursive formulae of the orthogonal irreducible decompositions for three-dimensional generic tensors of any finite orders.

Denote the generic element of $\mathcal{S}^{\otimes n}$ by $\mathbf{S}^{(n)}$. For instance, the photo-elastic tensor $\mathbf{M}$ is $\mathbf{S}^{(2)}$. It is well known that, in both two and three dimensions, the generic second order symmetric tensor $\mathbf{S}$ can be expressed in the unified form

$$\mathbf{S} = \alpha \mathbf{1} + \mathbf{D}^{(2)}. \tag{27}$$

Recalling the forms (15) and (16) for the sets of eigenvalues and characters of $\mathbf{R}^{\times 2}|_{\mathcal{D}}$, we obtain the sets and characters of $\chi(\mathbf{R}^{\times 2}|_{\mathcal{S}})$, respectively, as shown below

$$\{1, \mathrm{e}^{\pm \mathrm{i}2\theta}\}, \quad \chi(\mathbf{R}^{\times 2}|_{\mathcal{S}}) = 1 + 2 \cos 2\theta \tag{28}$$

in two dimensions, and

$$\{1, 1, \mathrm{e}^{\pm \mathrm{i}\theta}, \mathrm{e}^{\pm \mathrm{i}2\theta}\}, \quad \chi(\mathbf{R}^{\times 2}|_{\mathcal{S}}) = 2(1 + \cos \theta + \cos 2\theta) \tag{29}$$

in three dimensions.

Denote the coefficients of $\mathrm{e}^{\mathrm{i}s\theta}$ in the expansions of $(1 + \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})^n$ and of $(2 + \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta})^n$ by $\left[{n \atop s}\right]$ and $\left\{{2n \atop s}\right\}$, respectively. From (28), (29) and Theorem 3 we obtain the following result.

PROPOSITION 7. *The irreducible decomposition of the $2n$th-order tensor $\mathbf{S}^{(n)}$ in $\mathcal{E}$ as the generic nth-order tensor in $\mathcal{S}$ takes the form*

$$\mathbf{S}^{(n)} = j_0^{2n}\alpha \dot{+} j_1^{2n}\mathbf{v} \dot{+} j_2^{2n}\mathbf{D}^{(2)} \dot{+} \cdots \dot{+} j_{2n}^{2n}\mathbf{D}^{(2n)},$$

with, in two dimensions

$$j_s^{2n} = \begin{cases} 0, & \textit{if } s = \textit{odd}, \\ \begin{bmatrix} n \\ s/2 \end{bmatrix}, & \textit{if } s = \textit{even}, \end{cases} \qquad j_0^{2n} + 2\sum_{s=1}^{2n} j_s^{2n} = 3^n, \tag{30}$$

and in three dimensions

$$j_s^{2n} = \begin{Bmatrix} 2n \\ s \end{Bmatrix} - \begin{Bmatrix} 2n \\ s+1 \end{Bmatrix}, \qquad \sum_{s=0}^{2n}(1 + 2s)j_s^{2n} = 6^n. \tag{31}$$

Table 1 lists the nonzero numbers j_s^{2n} up to $n = 4$. For example, from this table we read that the irreducible decomposition of the $3D$ generic tensor $\mathbf{S}^{(2)}$ contains two scalars, one vector,

Table 1. The numbers of independent deviatoric tensors in the irreducible decompositions of $\mathbf{S}^{(2n)}$.

	2D					3D								
n	j_0^{2n}	j_2^{2n}	j_4^{2n}	j_6^{2n}	j_8^{2n}	j_0^{2n}	j_1^{2n}	j_2^{2n}	j_3^{2n}	j_4^{2n}	j_5^{2n}	j_6^{2n}	j_7^{2n}	j_8^{2n}
0	1					1								
1	1	1				1		1						
2	3	2	1			2	1	3	1	1				
3	7	6	3	1		5	6	11	7	6	2	1		
4	19	16	10	4	1	16	30	46	39	33	18	10	3	1

three second-order, one third-order and one fourth-order deviatoric tensors. This confirms the corresponding decomposition in the work of Zou *et al*.

4.2. Recursive formulae for physical tensors in types $\mathscr{S} \otimes \mathscr{D}^{(n)}$

Suppose that we have already the irreducible decomposition of $\mathbf{S}^{(n)} \in \mathscr{S}^{\otimes n}$ for some positive integer n and now aim to recursively obtain the irreducible decomposition of $\mathbf{S}^{(n+1)}$. This can be done in a way similar to that shown in the work of Zou *et al*., that is, it can be done by first considering the general elements $\hat{\mathbf{G}}^{(s+2)} \in \mathscr{S} \otimes \mathscr{D}^{(s)}$ for $s = 0, 1, \ldots, n$. In two dimensions, noting the character formulae

$$\chi(\mathbf{R}^{\times(2)}|_{\mathscr{S}\otimes\mathscr{D}^{(0)}}) = 1 + 2\cos 2\theta,$$
$$\chi(\mathbf{R}^{\times(3)}|_{\mathscr{S}\otimes\mathscr{D}^{(1)}}) = 2(1 + 2\cos 2\theta)\cos\theta = 2[2\cos\theta + \cos 2\theta],$$
$$\chi(\mathbf{R}^{\times(s+2)}|_{\mathscr{S}\otimes\mathscr{D}^{(s)}}) = 2(1 + 2\cos 2\theta)\cos s\theta = 2[\cos(s-2)\theta + \cos s\theta + \cos(s+2)\theta].$$

corresponding to $s = 0$, $s = 1$ and $s \geqslant 2$ and recalling Theorem 3, we obtain

$$\hat{\mathbf{G}}^{(2)} = \mathbf{D}^{(0)} \dotplus \mathbf{D}^{(2)}, \qquad \hat{\mathbf{G}}^{(4)} = 2\mathbf{D}^{(0)} \dotplus \mathbf{D}^{(2)} \dotplus \mathbf{D}^{(4)},$$
$$\hat{\mathbf{G}}^{(3)} = 2\mathbf{D}^{(1)} \dotplus \mathbf{D}^{(3)}, \qquad \hat{\mathbf{G}}^{(s+2)} = \mathbf{D}^{(s-2)} \dotplus \mathbf{D}^{(s)} \dotplus \mathbf{D}^{(s+2)}. \tag{32}$$

corresponding to $s = 0$, $s = 1$, $s = 2$ and $s > 2$. This is very similar to the brief form of the irreducible decomposition $(26)_2$ of $\mathbf{G}$ in three dimensions, see (39) of Zou *et al*. [6]. This recursive formula also yields the number formula (30) by recalling Theorem 3.

It can be seen that $\mathbf{1}\otimes\mathbf{D}^{(s)}$ or $\delta_{kl}D_{i_1\cdots i_s}$ is one proper $(s+2)$th order $\mathbf{D}^{(s)}$-induced irreducible tensor in the type $\mathscr{S} \otimes \mathscr{D}^{(s)}$. The $\mathbf{D}^{(s-2)}$-induced one can be expressed as a linear combination of

$$\delta_{kl}\delta_{\hat{i}_1\hat{i}_2}D_{\hat{i}_3\cdots\hat{i}_s}, \qquad \delta_{k\hat{i}_1}\delta_{l\hat{i}_2}D_{i_3\cdots i_s},$$
$$\delta_{\hat{i}_1\hat{i}_2}(\delta_{k\hat{i}_3}D_{\hat{i}_4\cdots\hat{i}_s l} + \delta_{l\hat{i}_3}D_{\hat{i}_4\cdots\hat{i}_s k}), \qquad \delta_{\hat{i}_1\hat{i}_2}\delta_{\hat{i}_3\hat{i}_4}D_{\hat{i}_5\cdots\hat{i}_s kl}.$$

where $T_{\hat{i}_1\cdots\hat{i}_p j\cdots k}$ denotes the symmetrization of $T_{i_1\cdots i_p j\cdots k}$ on the indices $(i_1\cdots i_p)$, which is defined as the sum of all the $p!$ permutations of $T_{i_1\cdots i_p j\cdots k}$ on $(i_1\cdots i_p)$, and then divided by $p!$. For instance,

$$T_{\hat{i}jkl\hat{m}} = \frac{1}{2!}(T_{ijklm} + T_{mjkli}), \qquad T_{\hat{i}\hat{j}\hat{k}} = \frac{1}{3!}(T_{ijk} + T_{jki} + T_{kij} + T_{jik} + T_{kji} + T_{ikj}).$$

Therefore, after some simple calculations, we can express (32) in details as

$$\hat{G}^{(2)}_{kl} = D_{kl} + \delta_{kl}\alpha, \tag{33}$$

$$\hat{G}^{(3)}_{kli_1} = D_{kli_1} + \delta_{kl}u_{i_1} + [\delta_{ki_1}v_l + \delta_{li_1}v_k - \delta_{kl}v_{i_1}], \tag{34}$$

$$\begin{aligned}\hat{G}^{(4)}_{kli_1i_2} \; = \; & D_{kli_1i_2} + \delta_{kl}D_{i_1i_2} + [\delta_{ki_1}\delta_{li_2} + \delta_{li_1}\delta_{ki_2} - \delta_{kl}\delta_{i_1i_2}]\alpha \\ & + [\delta_{ki_1}\varepsilon_{li_2} + \delta_{li_1}\varepsilon_{ki_2} + \varepsilon_{ki_1}\delta_{li_2} + \varepsilon_{li_1}\delta_{ki_2}]\beta,\end{aligned} \tag{35}$$

$$\begin{aligned}\hat{G}^{(s+2)}_{kli_1\ldots i_s} \; = \; & D_{kli_1\ldots i_s} + \delta_{kl}D_{i_1\cdots i_s} + \delta_{\hat{i}_1\hat{i}_2}\Big[\delta_{kl}D_{\hat{i}_3\cdots\hat{i}_s} + (s-2)(\delta_{k\hat{i}_3}D_{\hat{i}_4\cdots\hat{i}_s l} + \delta_{l\hat{i}_3}D_{\hat{i}_4\cdots\hat{i}_s k}) \\ & - \frac{(s-2)(s-3)}{4}\delta_{\hat{i}_3\hat{i}_4}D_{\hat{i}_5\cdots\hat{i}_s kl}\Big] - 2(s-1)\delta_{k\hat{i}_1}\delta_{l\hat{i}_2}D_{\hat{i}_3\cdots\hat{i}_s}.\end{aligned} \tag{36}$$

In three dimensions, similar to (32) we can get the following result for $s > 1$

$$\hat{\mathbf{G}}^{(s+2)} = \mathbf{D}^{(s-2)} \dotplus \mathbf{D}^{(s-1)} \dotplus 2\mathbf{D}^{(s)} \dotplus \mathbf{D}^{(s+1)} \dotplus \mathbf{D}^{(s+2)}. \tag{37}$$

4.2.1. *Physical tensors in types* sym $\mathcal{E}^{\otimes n}$ *and* sym $\mathcal{S}^{\otimes n}$

For any positive integer n, the completely symmetrization of $\mathbf{T}^{(n)} \in \mathcal{E}^{\otimes n}$ is denoted by sym $\mathbf{T}^{(n)}$, and the corresponding tensor subspace is denoted by sym $\mathcal{E}^{\otimes n}$. For instance, $\mathcal{S} =$ sym $\mathcal{E}^{\otimes 2}$. Similarly, as an nth-order tensor in $\mathcal{S}$, the complete symmetrization of $\mathbf{S}^{(n)} \in \mathcal{S}^{\otimes n}$ is denoted by sym $\mathbf{S}^{(n)}$, and the corresponding tensor subspace is denoted by sym $\mathcal{S}^{\otimes n}$. Thus, the elastic tensor $\mathbf{E}$ belongs to sym $\mathcal{S}^{\otimes 2}$. To calculate the characters of $\mathbf{R}^{\times n}$ and $\mathbf{R}^{\times 2n}$ as restrictions on sym $\mathcal{E}^{\otimes n}$ and sym $\mathcal{S}^{\otimes n}$, respectively, we give, but will omit the proof, the following proposition.

PROPOSITION 8. *Consider a tensor subspace $\mathcal{M}^{(r)}$ of rth-order and the nth-order tensor space $\mathcal{M}^{\otimes n}_{(r)}$, say in $\mathcal{M}^{(r)}$. Suppose that the eigenvalue set of $\mathbf{R}^{\times r}|_{\mathcal{M}^{(r)}}$ consists of c_0-fold 1, c_1-fold the pair $\mathrm{e}^{\pm \mathrm{i}\theta}$, c_2-fold the pair $\mathrm{e}^{\pm \mathrm{i}2\theta}, \ldots,$ and c_r-fold the pair $\mathrm{e}^{\pm \mathrm{i}r\theta}$. Introduce the generating function G*

$$G(U; x, y) = \sum_{I=1}^{c_0} x_{0I} + \sum_{I=1}^{c_1}(x_{1I}U + y_{1I}U^{-1}) + \cdots + \sum_{I=1}^{c_r}(x_{rI}U^r + y_{rI}U^{-r}), \tag{38}$$

and express the expansion of $G(U; x, y)^n$ into the form

$$\begin{aligned}G^n \; = \; & \sum_{J=1}^{\chi_0^{nr}} a_{0J}X_{0J} + \sum_{J=1}^{\chi_1^{nr}}(a_{1J}X_{1J}U + b_{1J}Y_{1J}U^{-1}) \\ & + \cdots + \sum_{J=1}^{\chi_{nr}^{nr}}(a_{(nr)J}X_{(nr)J}U^{nr} + b_{(nr)J}Y_{(nr)J}U^{-nr}),\end{aligned} \tag{39}$$

where X_{qJ} and Y_{qJ} are multiplies of x_I^r and/or y_K^t, and a_{qJ} and b_{qJ} are constants determined by $\{c_0, c_1, \ldots, c_r\}$. Then, the set of eigenvalues of the restriction $\mathbf{R}^{\times nr}|_{\mathrm{sym}\mathscr{M}_{(r)}^{\otimes n}}$ consists of χ_0^{nr}-fold 1, χ_1^{nr}-fold the pair $\mathrm{e}^{\pm\mathrm{i}\theta}$, S_2^n-fold the pair $\mathrm{e}^{\pm\mathrm{i}2\theta}, \ldots$, and χ_{nr}^{nr}-fold the pair $\mathrm{e}^{\pm\mathrm{i}nr\theta}$. The character is

$$\chi(\mathbf{R}^{\times nr}|_{\mathrm{sym}\mathscr{M}_{(r)}^{\otimes n}}) = \chi_0^{nr} + 2(\chi_1^{nr}\cos\theta + \chi_2^{nr}\cos 2\theta + \cdots + \chi_{nr}^{nr}\cos nr\theta). \tag{40}$$

For example, recalling (29), we observe that the eigenvalue set of $\mathbf{R}^{\times 2}|_{\mathscr{S}}$ in three dimensions is $\{1, 1, \mathrm{e}^{\pm\mathrm{i}\theta}, \mathrm{e}^{\pm\mathrm{i}2\theta}\}$. Therefore, the generating function in (38) takes the form

$$G = x_1 + x_2 + y_1U + y_2U^{-1} + z_1U^2 + z_2U^{-2}.$$

We have

$$\begin{aligned} G^2 \;=\; & (x_1^2 + x_2^2 + 2x_1x_2 + 2y_1y_2 + 2z_1z_2) \\ & +(2x_1y_1 + 2x_2y_1 + 2y_2z_1)U + (2x_1y_2 + 2x_2y_2 + 2y_1z_2)U^{-1} \\ & +(y_1^2 + 2x_1z_1 + 2x_2z_1)U^2 + (y_2^2 + 2x_2z_2 + 2x_1z_2)U^{-2} \\ & +2y_1z_1U^3 + 2y_2z_2U^{-3} + z_1^2U^4 + z_2^2U^{-4}. \end{aligned}$$

Thus, from Proposition 8 we know that the eigenvalue set of $\mathbf{R}^{\times 4}|_{\mathrm{sym}\,\mathscr{S}^{\otimes 2}}$ in three dimensions consists of five-fold 1, three-fold $\mathrm{e}^{\pm\mathrm{i}\theta}$, three-fold $\mathrm{e}^{\pm\mathrm{i}2\theta}$, one-fold $\mathrm{e}^{\pm\mathrm{i}3\theta}$, and one-fold $\mathrm{e}^{\pm\mathrm{i}4\theta}$. We obtain the character of $\mathbf{R}^{\times 4}|_{\mathrm{sym}\,\mathscr{S}^{\otimes 2}}$ and then the irreducible decomposition of $\mathrm{sym}\,\mathbf{S}^{(2)}$ (by applying Theorem 8) as follows

$$\chi(\mathbf{R}^{\times 4}|_{\mathrm{sym}\,\mathscr{S}^{\otimes 2}}) = 5 + 2(3\cos\theta + 3\cos 2\theta + \cos 3\theta + \cos 4\theta),$$

$$\mathrm{sym}\,\mathbf{S}^{(2)} = 2\alpha \dotplus 2\mathbf{D}^{(2)} \dotplus \mathbf{D}^{(4)}.$$

The last equation shows that the irreducible decomposition of a three-dimensional generic elastic tensor $\mathbf{E} = \mathrm{sym}\,\mathbf{S}^{(2)}$ contains independent two scalars, two second-order and one fourth-order deviatoric tensors. This coincides with the known result in the literature (*cf.* [3, 7]).

As an application of the recursive formulae from (33) to (36), we deduce the irreducible decompositions of $\mathrm{sym}\,\mathbf{S}^{(2)}$ and $\mathrm{sym}\,\mathbf{S}^{(3)}$ in two dimensions as follows. Using (33) and (35), we immediately obtain

$$S_{ijkl}^{(2)} = \delta_{ij}\delta_{kl}\alpha^1 + K_{ijkl}\alpha^2 + (\delta_{i\hat{k}}\varepsilon_{j\hat{l}} + \delta_{j\hat{l}}\varepsilon_{i\hat{k}})\beta^1 + \delta_{ij}D_{kl}^1 + \delta_{kl}D_{ij}^2 + D_{ijkl}, \tag{41}$$

$$\mathrm{sym}\,S_{ijkl}^{(2)} = \alpha^1\delta_{ij}\delta_{kl} + \alpha^2K_{ijkl} + \delta_{ij}D_{kl} + \delta_{kl}D_{ij} + D_{ijkl}, \tag{42}$$

where

$$K_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}.$$

Basing on (42) and (36), a tensor $H^{(6)} \in \mathscr{S} \otimes \mathrm{sym}\,\mathscr{S}^{\otimes 2}$ can be written as

$$H_{ijklmn}^{(6)} = \hat{G}_{ijklmn} + \delta_{kl}\hat{G}_{ijmn} + \delta_{mn}\hat{G}_{ijkl} + K_{klmn}\hat{G}_{ij}^1 + \delta_{kl}\delta_{mn}\hat{G}_{ij}^2, \tag{43}$$

Table 2. The numbers of independent deviatoric tensors in the irreducible decompositions of sym $\mathbf{S}^{(2n)}$.

n	$2D$						$3D$							
	j_0^{2n}	j_2^{2n}	j_4^{2n}	j_6^{2n}	j_8^{2n}	$\binom{n+2}{2}$	j_0^{2n}	j_2^{2n}	j_3^{2n}	j_4^{2n}	j_5^{2n}	j_6^{2n}	j_8^{2n}	$\binom{n+5}{5}$
0	1					1	1							1
1	1	1				3	1	1						6
2	2	1	1			6	2	2		1				21
3	2	2	1	1		10	3	3	1	2		1		56
4	3	2	2	1	1	15	4	5	1	4	1	2	1	126

where

$$\hat{G}^1_{ij} = D^3_{ij} + \alpha^2\delta_{ij}, \qquad \hat{G}^2_{ij} = D^4_{ij} + \alpha^3\delta_{ij},$$

$$\hat{G}_{ijmn} = \alpha^1 K_{ijmn} + \beta^1(\delta_{i\hat{m}}\varepsilon_{j\hat{n}} + \varepsilon_{i\hat{m}}\delta_{j\hat{n}}) + D^2_{ijmn} + \delta_{ij}D^2_{mn},$$

$$\hat{G}_{ijklmn} = (\delta_{ij}D^1_{\hat{m}\hat{n}} - \tfrac{1}{2}\delta_{\hat{m}\hat{n}}D^1_{ij} + 2\delta_{i\hat{m}}D^1_{\hat{n}j} + 2\delta_{j\hat{m}}D^1_{\hat{n}i})\delta_{\hat{k}\hat{l}} - 6\delta_{i\hat{k}}\delta_{j\hat{l}}D^1_{\hat{m}\hat{n}} + \delta_{ij}D^1_{klmn} + D_{ijklmn}.$$

Using the relations (*cf.* Zou *et al.*, [6])

$$\varepsilon_{is}D_{sjk\ldots l} = \varepsilon_{js}D_{sik\ldots l}, \qquad \delta_{ij}D_{kl} + \delta_{kl}D_{ij} = \delta_{ik}D_{jl} + \delta_{jl}D_{ik},$$

from (43), we can obtain the orthogonal irreducible decomposition of sym $\mathbf{S}^{(3)}$ in the following form

$$\begin{aligned}
\operatorname{sym} S^{(3)}_{ijklmn} \; = \; & \{\alpha^1\delta_{ij}\delta_{kl}\delta_{mn} + \alpha^2(K_{ijkl}\delta_{mn} + K_{klmn}\delta_{ij} + K_{mnij}\delta_{kl})\} \\
& + (\delta_{ij}\delta_{kl}D^1_{mn} + \delta_{kl}\delta_{mn}D^1_{ij} + \delta_{mn}\delta_{ij}D^1_{kl}) \\
& + (K_{ijkl}D^2_{mn} + K_{klmn}D^2_{ij} + K_{mnij}D^2_{kl}) \\
& + (\delta_{ij}D_{klmn} + \delta_{kl}D_{ijmn} + \delta_{mn}D_{ijkl}) + D_{ijklmn}.
\end{aligned} \tag{44}$$

Finally, if we give up the requirement of the orthogonality, (44) can be further expressed in the form

$$\begin{aligned}
\operatorname{sym} S^{(3)}_{ijklmn} \; = \; & \{\alpha^1\delta_{ij}\delta_{kl}\delta_{mn} + \alpha^2(I_{ijkl}\delta_{mn} + I_{klmn}\delta_{ij} + I_{mnij}\delta_{kl})\} \\
& + (\delta_{ij}\delta_{kl}D^1_{mn} + \delta_{kl}\delta_{mn}D^1_{ij} + \delta_{mn}\delta_{ij}D^1_{kl}) \\
& + (I_{ijkl}D^2_{mn} + I_{klmn}D^2_{ij} + I_{mnij}D^2_{kl}) \\
& + (\delta_{ij}D_{klmn} + \delta_{kl}D_{ijmn} + \delta_{mn}D_{ijkl}) + D_{ijklmn}.
\end{aligned} \tag{45}$$

In Table 2, the numbers of independent deviatoric tensors for sym $\mathbf{S}^{(n)}$ up to $n = 4$ are listed, which are obtained by the use of Proposition 8 and Theorem 3. These results are confirmed by the right total numbers $\binom{n+2}{2}$ in two dimensions and $\binom{n+5}{5}$ in three dimensions, respectively, of the independent components of sym $\mathbf{S}^{(n)}$. For example, it is well known that the total number of independent components of a $3D$ elastic tensor $\mathbf{E} = \operatorname{sym}\mathbf{S}^{(2)}$ is equal to 21. From Table 2 we see that the two scalars, two second-order and one fourth-order deviatoric tensors involved in the irreducible decomposition of sym $\mathbf{S}^{(2)}$ contribute just 21 independent components.

We conclude this section by the irreducible decomposition of generic completely symmetric tensors sym $\mathbf{T}$.

PROPOSITION 9. *In both two and three dimensions, the decompositions of completely symmetric tensors take the following unified forms*

$$\begin{aligned} \operatorname{sym}\mathbf{T}^{(2n)} &= \alpha \dotplus \mathbf{D}^{(2)} \dotplus \mathbf{D}^{(4)} \dotplus \cdots \dotplus \mathbf{D}^{(n)} = \operatorname{sym}\sum_{r=0}^{n} \mathbf{1}^{\otimes(n-r)} \otimes \mathbf{D}^{(2r)}, \\ \operatorname{sym}\mathbf{T}^{(2n+1)} &= \mathbf{v} \dotplus \mathbf{D}^{(3)} \dotplus \mathbf{D}^{(5)} \dotplus \cdots \dotplus \mathbf{D}^{(n)} = \operatorname{sym}\sum_{r=0}^{n} \mathbf{1}^{\otimes(n-r)} \otimes \mathbf{D}^{(2r+1)}. \end{aligned} \tag{46}$$

In (46), $\mathbf{1}^{\otimes s}$ denotes the sth tensor power, *e.g.* $\mathbf{1}^{\otimes 2} = \mathbf{1}\otimes\mathbf{1}$ and $\mathbf{1}^{\otimes 3} = \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$. The number of independent deviatoric tensors in (46) is calculated according to Proposition 8 and Theorem 3, and the forms of $\mathbf{D}^{(s)}$-induced completely symmetric irreducible tensors are obtained from the observation that ε is skew-symmetric, so that can not be effective.

5. Concluding remarks

In this paper tensors in tensor subspaces have received the name physical tensors. We developed some simple and practical techniques to obtain irreducible decompositions of physical tensors of high orders, particularly to provide a fast procedure for giving the number of independent deviatoric tensors contained in the irreducible decomposition of physical tensors. As for applications, a number of detailed irreducible decompositions of physical tensors of high orders are first constructed. These results should be of increasing interest and importance, since more and more research fields (such as the strain-gradient theory, second-order elasticity theory, damage mechanics with a fourth-order damage tensor, etc.) involve high-order physical tensors. It is emphasized that irreducible decompositions of high-order tensors enable easier and more intuitionistic physical understanding and characteristics of these tensors, and may even become a necessity in formulating a physical or constitutive equation which involves these high-order tensors.

Appendix A. Spherical harmonics

In the group representation theory it is known that deviatoric tensors can be one-to-one connected with spherical harmonics. For making this paper self-contained, we derive the relationships among deviatoric tensors, spherical harmonics and Fourier expansions in N-dimensional basic Euclidean space $\mathcal{E}$. In the following the summation convention for respected indices is not adopted.

Denote by $\mathcal{P}_m$ the linear space of all homogeneous polynomials in the coordinate variables x_i of degree m, for example, $x_1^6\, x_2^2\, x_3^9 \in \mathcal{P}_{17}$. Denote by $\Delta = \sum_{i=1}^{N} \partial^2/\partial x_i\,\partial x_i$ the Laplace operator in the basic N-dimensional Euclidean space $\mathcal{E}$. The linear space $\{p(\mathbf{x}) \in \mathcal{P}_m\colon \Delta p(\mathbf{x}) = 0\}$ is called the space of harmonic polynomials of degree m, and the restricting forms of harmonic polynomials of degree m to the unit sphere in $\mathcal{E}$ are called *spherical harmonics* of degree m. We denote by $\mathcal{H}_m$ the linear space of all spherical harmonics of degree m. It is well known (*cf.* [16, pp. 302–309], [17, pp. 88–89]) that the linear space $\mathcal{H}_m$ and the mth order irreducible tensor space $\mathcal{D}^{(m)}$ have the same dimensions

$$D_m = \dim \mathcal{H}_m = \dim \mathcal{D}^{(m)}.$$

In general, the spaces $\mathcal{H}_m$ and $\mathcal{H}_n$ for different nonnegative integers m and n are mutually orthogonal in the following inner product sense

$$(f, g) = \oint f(\mathbf{n})g(\mathbf{n})\,\mathrm{d}\mathbf{n} = 0, \quad \text{for any } f \in \mathcal{H}_m,\ g \in \mathcal{H}_n, \tag{A1}$$

where $\mathbf{n}$ denotes the generic unit vector and the integration is over the unit sphere.

It is well known that any square-integrable scalar function $F(\mathbf{n})$ on the unit sphere can be expanded into an absolutely convergent Fourier series in the form

$$F(\mathbf{n}) = F_0 + \sum_{m=1}^{\infty} F_m(\mathbf{n}), \quad F_m(\mathbf{n}) \in \mathcal{H}_m.$$

Denoting by Ω the area of the unit sphere in $\mathcal{E}$, we have

$$F_0 = \Omega^{-1} \oint F(\mathbf{n})\,\mathrm{d}\mathbf{n}.$$

To express $F_m(\mathbf{n})$, let us introduce an orthogonal basis, say $\{Y_{m,J}(\mathbf{n})\colon J = 1, \ldots, D_m\}$, of $\mathcal{H}_m$. For instance, the sets

$$\{\cos m\phi, \sin m\phi\},$$

$$\{P_{m,0}(\cos\theta), P_{m,1}(\cos\theta)\cos\phi, P_{m,1}(\cos\theta)\sin\phi, \ldots,$$
$$P_{m,m}(\cos\theta)\cos m\phi, P_{m,m}(\cos\theta)\sin m\phi\},$$

constitute orthogonal bases of $\mathcal{H}_m$ in two and three dimensions, respectively, where

$$\mathbf{n} = (\cos\phi, \sin\phi) \quad \textit{or} \quad \mathbf{n} = (\cos\theta\ \cos\phi, \cos\theta\ \sin\phi, \sin\phi)$$

denotes the polar or spherical polar coordinate representation of the two- or three-dimensional unit vector $\mathbf{n}$, and $P_{m,r}(y)$ are the Legendre and associated Legendre functions that can be expressed in terms of the Rodrigues forms as follows

$$P_{m,r}(y) = \frac{(1-y^2)^{r/2}}{2^m m!} \frac{\mathrm{d}^{m+r}}{\mathrm{d}y^{m+r}}(y^2-1)^m.$$

By definition, we have

$$F_m(\mathbf{n}) = \sum_{J=1}^{D_m} a_{m,J} Y_{m,J}(\mathbf{n}), \quad a_{m,J} = (Y_{m,J}, Y_{m,J})^{-1} \oint F(\mathbf{n}) Y_{m,J}(\mathbf{n})\,\mathrm{d}\mathbf{n}.$$

A one-to-one correspondence relation between the generic spherical harmonics $F_m(\mathbf{n})$ of degree m and the generic mth order deviatoric tensor $\mathbf{D}^{(m)}$ can be established in the following form

$$F_m(\mathbf{n}) = \mathbf{D}^{(m)} \circ \mathbf{n}^{\otimes m} = \sum_{i_1 i_2 \ldots i_m} D_{i_1 i_2 \ldots i_m} n_{i_1} n_{i_2} \ldots n_{i_m},$$

where

$$\mathbf{D}^{(m)} = \sum_{J=1}^{D_m} a_{m,J}\mathbf{g}_{m,J}, \qquad Y_{m,J}(\mathbf{n}) = \mathbf{g}_{m,J} \circ \mathbf{n}^{\otimes m}, \quad \mathbf{g}_{m,J} = \Omega_m^{-1} \oint Y_{m,J}(\mathbf{n})\mathbf{n}^{\otimes m}\,\mathrm{d}\mathbf{n},$$

with

$$\Omega_m = \frac{m!\Omega}{N(N+2)\cdots(N+2m-2)}, \quad N = \dim \mathcal{E}.$$

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